1 a Even though there are no restrictions, the first digit cannot be 0. Therefore, there are six choices for the first digit. There are then six choices for the second digit (including 0), five for the third and so on. Using the multiplication principle, there are

$$6 \times 6 \times 5 \times 4 \times 3 = 2160$$

different numbers.

b If the number is divisible by 10 then the last digit must be 0. Therefore, there is only one choice for this digit. There are then six choices for the first digit, five for the second and so on. Using the multiplication principle, there are

$$6 \times 5 \times 4 \times 3 \times 1 = 360$$

different numbers.

c If the number is odd, then the last digit must be one of three options: $1, 3$ or 5. The first digit cannot be 0, and obviously can't be equal to the last digit. Therefore, there are five choices for the first digit. There are then five choices for the second digit (including 0), four for the third and so on. Using the multiplication principle, there are

$$5 \times 5 \times 4 \times 3 \times 3 = 900$$

different numbers.

d There are a total 2160 numbers, of which 900 are odd. The remaining $2160 - 900 = 1260$ will be even.

2 a There are eight workers in total, from which four are to be selected. This can be done in

$$^8C_4 = 70$$

different ways.

b We must select two of three men and two of five women. Using the multiplication principle this can be done in

$$^3C_2 \times ^5C_2 = 30$$

different ways.

c If the group must contain Mike and Sonia then we need only select two more workers from the six that remain. This can be done in

$$^6C_2 = 15$$

different ways.

d If the group cannot contain both Mike and Sonia, then we need only evaluate the total number of selections, then subtract those selections that contain both Mike and Sonia. This gives

$$70 - 15 = 55$$

different selections.

3 a There are six items in total, of which a group of three are alike and another group of three are alike. These can be arranged in

$$\frac{6!}{3! \times 3!} = 20$$

different ways.

b There must be at least one red flag between each black flag. Denote black and red flags by the letters B and R respectively. Then consider the sequence BRBRB. This arrangement isolates the black flags using two red flags. The third red flag can be inserted anywhere, giving four different arrangements:

$$\text{RBRBRB}, \text{BRRBRB}, \text{BRBRRB}, \text{BRBRBR}.$$

c We list all of the possibilities in the table below. In the first two columns we write down the numbers of red and black flags respectively.

R	B	arrangements
1	0	1
0	1	1
0	2	1
2	0	1
1	1	2
3	0	1
0	3	1
1	2	3
2	1	3
1	3	4
3	1	4
2	2	6
2	3	10
3	2	10
3	3	20

This gives a total of 68 different arrangements.

4 a There are seven letters in total, of which a group of three Gs are alike and another group of two As are alike. These can be arranged in

$$\frac{7!}{3! \times 2!} = 420$$

different ways.

b There are three cases to consider, each of which gives the same number of arrangements.
Case 1: If the arrangement begins and ends with A then there are now just five letters to arrange, of which a group of three Gs are alike. These can be arranged in

$$\frac{5!}{3!} = 20$$

different ways.
Case 2: If the arrangement begins with A and ends with E then there are now just five letters to arrange, of which a group of three Gs are alike. These can be arranged in

$$\frac{5!}{3!} = 20$$

different ways.
Case 3: If the arrangement begins with E and ends with A then there are now just five letters to arrange, of which a group of three Gs are alike. These can be arranged in

$$\frac{5!}{3!} = 20$$

different ways
Therefore the total number of arrangements will be $20 + 20 + 20 = 60$.

c There are three cases to consider.
Case 1: If the arrangement begins and ends with a G then there are now just five letters to arrange, of which a group of two As are alike. These can be arranged in

$$\frac{5!}{2!} = 60$$

different ways.
Case 2: If the arrangement begins with B and ends with a G then there are now just five letters to arrange, of which a group of two Gs are alike and a group of two As are alike. These can be arranged in

$$\frac{5!}{2!2!} = 30$$

different ways.
Case 3: If the arrangement begins with a G and ends with B then there are now just five letters to arrange, of which a group of two Gs are alike and a group of two As are alike. These can be arranged in

$$\frac{5!}{2!2!} = 30$$

different ways.
Therefore the total number of arrangements will be $60 + 30 + 30 = 120$.

d We group together all of the vowels {A,A,E} and all of the consonants {B,G,G,G}. There are now two groups to arrange .This can be done in 2 ways. We then arrange within each group. The first group can be arranged in $\frac{3!}{2!} = 3$ different ways, and the second group can be arranged in $\frac{4!}{3!} = 4$ different ways. Using the multiplication principle, the total number of different arrangements will be,

$$2 \times 3 \times 4 = 24.$$

5 a There are many ways to answer this question, each giving the same answer.
Method 1: There are
$^{25}C_2 = 300$
ways of selecting two of twenty-five people to shake hands.
Method 2: The first person shakes hands with 24 others, the second with 23 and so on. This gives the total number of handshakes as

$$24 + 23 + \cdots + 1 = 300.$$

Method 3: Each of the 25 people shakes hands with 24 others, but this double counts each handshake. Therefore the total number of handshakes is

$$\frac{25 \times 24}{2} = 300.$$

b This question can be done by trial and error. Here's an algebraic solution. Suppose that there are n people in the first group and $25 - n$ people in the second group. Then,

$$\begin{aligned} {}^nC_2 +^{25-n} C_2 &= 150 \\ \frac{n!}{2!(n-2)!} + \frac{(25-n)!}{2!(23-n!} &= 150 \\ \frac{n(n-1)(n-2)!}{2!(n-2)!} + \frac{(25-n)(24-n)(23-n)!}{2!(23-n!} &= 150 \\ \frac{n(n-1)}{2} + \frac{(25-n)(24-n)}{2} &= 150 \\ n(n-1) + (25-n)(24-n) &= 300 \\ n^2 - 25n + 150 &= 0 \\ (n-10)(n-15) &= 0 \\ n &= 10, 15. \end{aligned}$$

Therefore, the number of people in each group is 10 and 15.

c If we tried to count the total number of handshakes then each of the 25 people shakes hands with exactly 3 others. This double counts each handshake, so the total number of handshakes is

$$\frac{25 \times 3}{2} = \frac{75}{2},$$

which not a whole number.

6 a Four points can be selected from twelve in ${}^{12}C_4 = 495$ ways.

b Two points can be selected from twelve in ${}^{12}C_2 = 66$ ways. From this, subtract the 6 pairs that are diametrically opposite. This gives a total of $66 - 6 = 60$.

c Pick any two vertices that are not diametrically opposite. These two points, and the two points that are diametrically opposite, will lie on a rectangle.

d Selecting any two points that are not diametrically opposite will define the edge of a rectangle as described in the previous question. This can be done in 60 ways. However, there are four edges that give the same rectangle. Therefore, the total number of rectangles will be $60 \div 4 = 15$.

e There are a total of 495 choices of 4 points, and of these 15 are rectangles. Therefore, the probability of selecting a rectangle is

$$\frac{15}{495} = \frac{1}{33}.$$

7 a If $a + b$ is even then either a and b are both odd or a and b are both even. If $b + c$ is odd then either b is odd and c is even or b is even and c is odd. Therefore, one of these two statement must be true:
Statement 1: a is odd and b is odd and c is even.
Statement 2: a is even and b is even and c is odd.
Therefore, we can't determine whether a, b or c are even or odd. For instance, the numbers $a = b = 1$ and $c = 2$ satisfy the given conditions, as do the numbers $a = b = 2$ and $c = 1$.

b If we additionally know that $a + b + c$ is even then the second statement above cannot be true, as $a + b + c$ would be odd. Therefore, the first statement must be true. Therefore, a is odd and b is odd and c is even.

8 a We first show that $a = 2$. If $a = 1$ then the left hand side is too large. If $a \geq 2$ then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Therefore, $c = 2$ and

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$

We now show that $b = 3$. If $b \geq 4$ then

$$\frac{1}{b} + \frac{1}{c} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore, $b = 3$ and

$$\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Therefore, $c = 6$. We have obtained just one set of values:

$$(a, b, c) = (2, 3, 6).$$

b We first show that $a = 1$. If $a \geq 2$ then the left hand side is too small since,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} < \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$$

Therefore, $a = 1$ so that we now require that

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 1.$$

We now show that $b = 2$. If $b \geq 3$ then

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{d}<\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1.$$

Therefore, $b=2$ so that we now require that

$$\frac{1}{c}+\frac{1}{d}>\frac{1}{2}.$$

We now show that $c=3$. If $c\geq 4$ then

$$\frac{1}{c}+\frac{1}{d}<\frac{1}{4}+\frac{1}{4}=\frac{1}{2}.$$

Therefore, $c=3$ so that we now require that

$$\frac{1}{d}>\frac{1}{2}-\frac{1}{3}=\frac{1}{6}.$$

Therefore, either $d=4$ or $d=5$. We have obtained just two sets of values:

$$(a,b,c,d)=(1,2,3,4) \text{ or } (a,b,c,d)=(1,2,3,5).$$

9 Suppose that $b>a$. Then

$$\begin{aligned}\frac{a+c}{b+c}-\frac{a}{b}&=\frac{b(a+c)}{b(b+c}-\frac{a(b+c)}{b(b+c)}\\&=\frac{b(a+c)-a(b+c)}{b(b+c)}\\&=\frac{ab+bc-ab-ac}{b(b+c)}\\&=\frac{bc-ac}{b(b+c)}\\&=\frac{c(b-a)}{b(b+c)}\\&>0\end{aligned}$$

Note that the last line follows from the fact that each term in the fraction is positive. Therefore,

$$\frac{a+c}{b+c}>\frac{a}{b},$$

as required.

10a Since $2^9=512<10^3$ and $2^{10}=1024>10^3$, the smallest such n will be 10.

b Since,

$$\begin{aligned}2^{100}&=(2^{10})^{10}\\&>(10^3)^{10}\\&=10^{30},\end{aligned}$$

we know that 2^{100} must have at least 31 digits.

c As there are at least 31 digits, and 10 different digits, there must be some digit that occurs at least 4 times.

11a Since the newspaper has 100 pages and each sheet includes 4 pages, the stack must contain $100\div 4=25$ sheets. The 25th sheet includes pages $49, 50, 51$ and 52.

b The least two numbers have increased by 6 from 1 and 2 to 7 and 8. The last two pages will decrease by 6 from 99 and 100 to 93 and 94.

c Suppose the newspaper is make up of n sheets of paper. Then the kth sheet of paper will include pages $2k-1, 2k, 4n-2k+1, 4n-2k+2$. The sum of these numbers is

$$2k-1+2k+4n-2k+1+4n-2k+2=8n+2.$$

Therefore, the sum of page numbers on each sheet depends only on the total number of sheets.

d From the previous question, we see that

$$\begin{aligned} 8n+2 &= 11+12+33+34 \\ 8n+2 &= 90 \\ 8n &= 88 \\ n &= 11. \end{aligned}$$

There are 11 sheets of paper. Therefore, there are $11 \times 4 = 44$ pages.

12a The smallest number of coins that Sam would need to do this is

$$0+1+2+3+4+5+6=21.$$

Sam has only 20 coins, so this is impossible.

b The calculation above shows that Sam would need 21 coins.

c Since

$$0+1+2+3+4+5+6+7+8+9=45,$$

Sam could fill 10 pockets with $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 coins. His last five coins could go in the pocket containing 9 coins. Each pocket would then have a different number of coins. We now show that it is impossible for him to fill more than 10 pockets with a different number of coins in each. Arrange these numbers from smallest to largest. The smallest eleven numbers are no less than $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ respectively, and the sum of these numbers is

$$0+1+2+3+4+5+6+7+8+9+10=55>50.$$

13a If the first digit is n and the second digit is 5 then the last two digits of its square will be 25 and the first two digits will be $n \times (n+1)$.

b Since $7 \times 8 = 56$, from the observed pattern we expect that $75^2 = 5625$. You can easily check that this is true.

c Each number is of the form $10n+5$. We square this number to obtain

$$\begin{aligned} (10n+5)^2 &= 100n^2+100n+25 \\ &= 100n(n+1)+25 \end{aligned}$$

This shows that the first two digits will be $n(n+1)$ and the last two digits will be 25.

14a Note that

$$1+2+\cdots+10=\frac{10 \times 11}{2}=55.$$

If the blocks could somehow be used to build two towers of the same height, then each would be $55 \div 2 = 27.5$ cm tall. This is impossible, as each block has an integer side length.

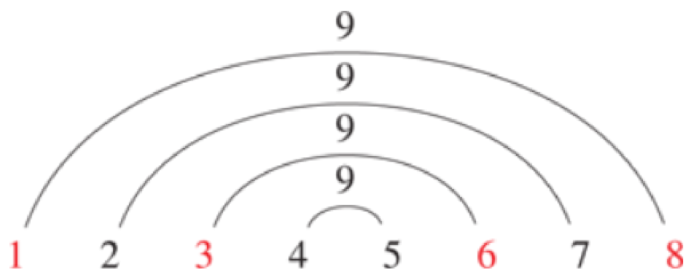
b If $n = 4k+1$ or $n = 4k+2$ then you cannot build two towers of the same height. First suppose $n = 4k+1$. Then note that

$$\begin{aligned} 1+2+\cdots+(4k+1) &= \frac{(4k+1)(4k+2)}{2} \\ &= (4k+1)(2k+1). \end{aligned}$$

Since this is odd, it is not divisible by 2. Similarly, if $n = 4k+2$ then,

$$\begin{aligned} 1+2+\cdots+(4k+2) &= \frac{(4k+2)(4k+3)}{2} \\ &= (2k+1)(4k+3). \end{aligned}$$

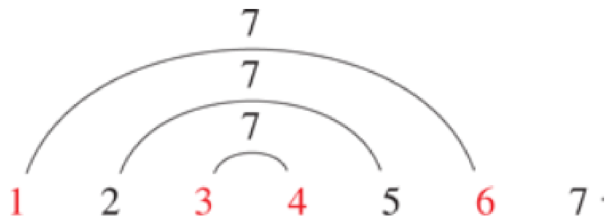
Since this is odd, it is not divisible by 2.
We will prove $n = 4k$ or $n = 4k-1$ then we can build two towers of the same height. When $n = 4k$, this is easy. We indicate how this can be done with an example that is easily generalised. Let $n = 8$. By pairing 1 with 8, then 2 with 7, etc., we see that each pair has the same sum.



If follows that we can make two towers whose heights are the same. For example,

$$1+3+6+8=2+4+5+7.$$

When $n=4k-1$, we do something similar. We indicate how this can be done with a an example that is easily generalised. Let $n=7$. By pairing 1 with 6, 2 with 5 etc., we see that each pair has the same sum, 7. Notice that 7 does not belong to a pair.



Once again, using this diagram we can make two towers whose heights are the same. For example,

$$1+3+4+6=2+5+7.$$

15a Suppose that a is odd and b is odd. Then $a=2k+1$ and $b=2m+1$ where $k,m\in\mathbb{Z}$. Therefore,

$$\begin{aligned}ab&=(2k+1)(2m+1)\\&=4k^2+2k+2m+1\\&=2(2k^2+k+m)+1\\&=2n+1,\ \text{where } n=2k^2+k+m\in\mathbb{Z}.\end{aligned}$$

We see that ab is odd.

b $\boxed{P(n)}$

If $n\in\mathbb{N}$ and a is odd then a^n is odd

$\boxed{P(1)}$

If $n=1$ then $a^1=a$ is odd, by assumption. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that a^k is odd.

$\boxed{P(k+1)}$

Since

$$a^{k+1}=a^k\times a$$

is the product of two odd numbers, a^{k+1} will be odd. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n\in\mathbb{N}$ by the principle of mathematical induction.

c Assume, on the contrary, that $3^{\frac{n}{m}}=2$ where $n,m\in\mathbb{N}$. Then raising both sides to the power of m gives,

$$\begin{aligned}\left(3^{\frac{n}{m}}\right)^m&=2^m\\3^n&=2^m\end{aligned}$$

We have proved that the left hand side is odd. However, the right hand side is even. This is a contradiction.

16a Expanding the left hand side gives

$$n^4+6n^3+11n^2+6n+1=a^2n^4+2abn^3+(2ac+b^2)n^2+2bcn+c^2.$$

We then equate coefficients. Since a is positive and $a^2=1$, clearly $a=1$. Likewise $c=1$. Finally, as $2bc=6$, we know that $b=3$.

b Let the four consecutive numbers be $n, n+1, n+2$ and $n+4$. Then when 1 is added to their product, we obtain

$$n(n+1)(n+2)(n+3)+1.$$

If we expand this expression, we obtain

$$n^4+6n^3+11n^2+6n+1.$$

From the previous question, we know that this is equal to

$$n^4+6n^3+11n^2+6n+1=(n^2+3n+1)^2.$$

c Let $n=5$ in the previous question, so that

$$\begin{aligned}5\times 6\times 7\times 8+1&=(5^2+3\times 5+1)^2\\&=41^2\end{aligned}$$

17a We first note that,

$QR=CB=a$

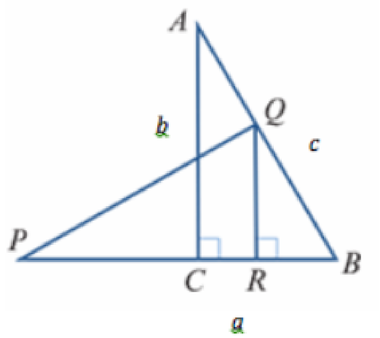
$PQ=AB=c$

$PR=AC=b$

$\angle PBQ=\angle CBA$ (common)

$\angle P=\angle A$ ($\triangle ABC\equiv\triangle PQR$)

$\therefore \triangle PBQ\sim\triangle ABC$



b $\dfrac{PB}{PQ}=\dfrac{AB}{AC}$

$\therefore PB=\dfrac{c^2}{b}$

c $\triangle QBR\sim\triangle ABC$ (AAA)

d $PB=PR+RB$

$PB=b+\dfrac{a^2}{b}$

e Use the expressions for PB obtained above.

$\dfrac{c^2}{b}=b+\dfrac{a^2}{b}$

$\therefore c^2=b^2+a^2$

18a Since $\angle TBO+\angle TAO=180°$, $TBOA$ is a cyclic quadrilateral.

b $\angle BPT=x°$

$\angle BCA=x°$ (corresponding)

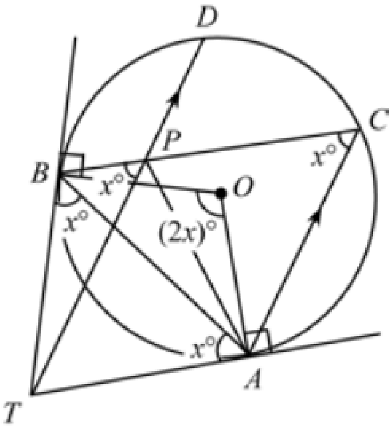
$\angle BOA=(2x)°$

(angle subtended at centre is twice the angle at the circumference)

$$\begin{aligned}\angle TAB&=\angle ACB\text{(alternate segment)}\\&=x°\end{aligned}$$

$\angle TBA=\angle AC$(alternate segment)

$= x^\circ$



19a $\angle GAF = \angle HAB$ common

$\dfrac{AB}{AF} = \dfrac{AH}{AG}$ construction

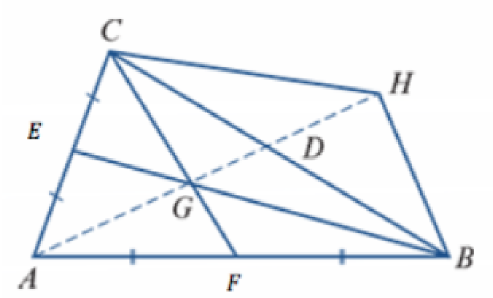
$\therefore \triangle AFG \sim \triangle ABH$

b $\angle GFA = \angle HBA$ $(\triangle AFG \sim \triangle ABH)$

$\therefore CG \parallel BH$ (corresponding angles equal)

c Show $\triangle AEG \sim \triangle ACH$ (Exactly the same as part a

$\therefore \angle AEG = \angle ACH$



$\therefore GB \parallel CH$

d $CG \parallel BH$ and $GB \parallel CH$

$\therefore GBHC$ is a parallelogram

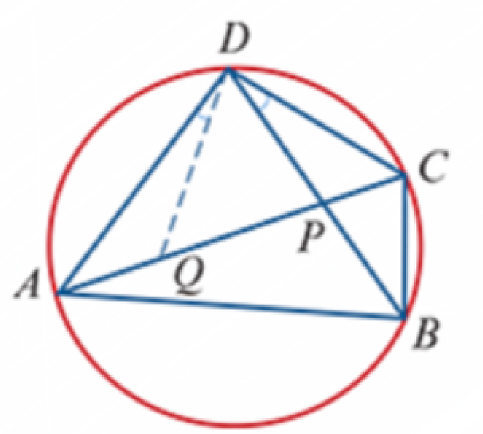
Diagonals of a parallelogram bisect eachother.

Hence $BD = DC$

20a i $\angle BDC = \angle ADQ$ (construction)

$\angle DAQ = \angle DBC$ (subtended by the same arc)

$\therefore \triangle ADQ \sim \triangle BDC$ (AAA)



ii $\angle QCD = \angle ABD$ (construction)

Let $\angle BDC = \angle ADQ = \theta$

$\angle QDB = 180^\circ - 2\theta$

$\therefore \angle QDC = 180^\circ - 2\theta + \theta = 180^\circ - \theta$

$\angle ADB = 180^\circ - 2\theta + \theta = 180^\circ - \theta$

$\therefore \triangle ADB \sim \triangle QDC$ (AAA)

iii From i $\dfrac{AQ}{BC} = \dfrac{AD}{BD}$

$\therefore AQ = \dfrac{BC \cdot AD}{BD}$

From ii $\dfrac{QC}{AB} = \dfrac{CD}{BD}$

$\therefore QC = \dfrac{AB \cdot CD}{BD}$

iv $AC = AQ + QC$

$\therefore AC = \dfrac{BC \cdot AD}{BD} + \dfrac{AB \cdot CD}{BD}$

$\therefore AC\dot{B}D = BC \cdot AD + AB \cdot CD$

The theorem is proved.

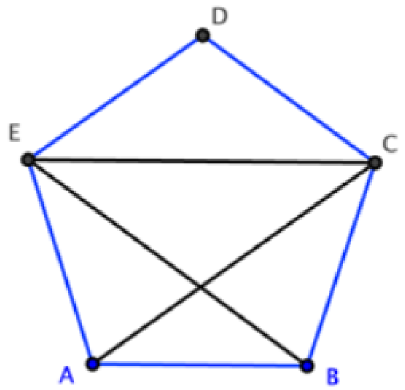
b In a rectangle $BC = AD$, $CD = AB$ and since the diagonal are of equal length $AC = BD$

$\therefore BC^2 + AB^2 = AC^2$

c Let $AB = BC = CD = DE = EA = 1$

Let x be tlength of a diagonal.

Apply Ptolemy's theorem to quadrilateral ABCE.



$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

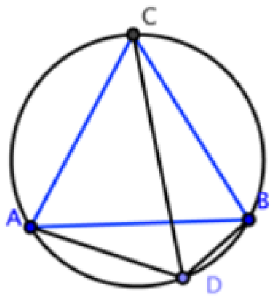
$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1 + \sqrt{5}}{2}$$

d Let $AB = BC = CA = 1$

Apply Ptolemy's theorem to quadrilateral ADBC.

$CD \times 1 = AD \times 1 + DB \times 1.$



$\therefore CD = AB + DB$